1. Exercises from 5.1

Today we're going to practice computing some line integrals.

Problem 1. *(Folland 5.1.1(a,b,c,d) Find the arclengths of the following paths*

In all of these examples, we use Theorem 5.11 on p. 220.

Part a) $\gamma(t) = (a\cos t, a\sin t, bt), t \in [0, 2\pi]$: The curve is a helix in $\mathbb{R}^3$ oriented about the z-axis.

$$\gamma'(t) = (-a\sin t, a\cos t, b) \Rightarrow |\gamma'(t)| = \sqrt{(-a\sin t)^2 + (b\cos t)^2 + b^2} = \sqrt{a^2+b^2}$$

So the arclength is:

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|\, dt = \int_0^{2\pi} \sqrt{a^2+b^2}\, dt = 2\pi\sqrt{a^2+b^2}$$

Part b) $\gamma(t) = (\frac{1}{3}t^3 - t, t^2), t \in [0, 2]$:

$$\gamma'(t) = (t^2-1, 2t) \Rightarrow |\gamma'(t)| = \sqrt{(t^2-1)^2 + 4t^2} = \sqrt{t^4+2t^2+1} = \sqrt{(t^2+1)^2} = t^2+1$$

So the arclength is:

$$L(\gamma) = \int_0^2 |\gamma'(t)|\, dt = \int_0^2 t^2 + 1\, dt = \left[\frac{1}{3}t^3 + t\right]\Big|_0^2 = \frac{14}{3}$$

Part c) $\gamma(t) = (\log t, 2t, t^2), t \in [1, e]$:

$$\gamma'(t) = \left(\frac{1}{t}, 2, 2t\right) \Rightarrow |\gamma'(t)| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \frac{2}{t}\sqrt{t^4 + t^2 + \frac{1}{4}} = \frac{2}{t}\left(t^2 + \frac{1}{2}\right) = 2t + \frac{1}{t}$$

So the arclength is:

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)|\, dt = \int_1^e 2t + \frac{1}{t}\, dt = \left[t^2 + \log t\right]\Big|_1^e = e^2 + 1 - 1 - 0 = e^2$$

Exercise 1.1. *Do part d*

Problem 2. *(Folland 5.1.4) Compute*

$$\int_C \sqrt{z}\, ds$$

Where C is the curve parameterized by $\gamma(t) = (2\cos t, 2\sin t, t^2)$ for $t \in [0, 2\pi]$.

Remember that in general when we are computing the integral of a scalar function along a curve,

$$\int_C f(x,y,z)\, ds = \int_{t_1}^{t_2} f(\gamma(t))\, |\gamma'(t)|\, dt$$

So to compute these kinds of integrals we need three things:

(1) $\gamma(t) : [t_1, t_2] \to \mathbb{R}^3$, a parameterization of the path C
(2) $\gamma'(t)$, the "velocity" of the curve at each point along the path
(3) Must know how to evaluate $f(x,y,z)$ at each point along $\gamma(t)$

In this problem, (1) is given to us. We can calculate (2):

$$\gamma'(t) = (-2\sin t, 2\cos t, 2t) \Rightarrow |\gamma'(t)| = \sqrt{4+4t^2}$$

And we can find (3) by evaluating along the path:

$$f(\gamma(t)) = \sqrt{z} \circ \gamma(t) = \sqrt{t^2} = t$$

Combining everything gives:

$$\int_C \sqrt{z}\, ds = \int_0^{2\pi} 2t\sqrt{1+t^2}\, dt = \int_0^{4\pi^2} \sqrt{1+u}\, du = \frac{2}{3}(1+u)^{3/2}\Big|_0^{4\pi^2} = \frac{2}{3}\left[(1+4\pi^2)^{3/2} - 1\right]$$

PROBLEM 3. *(Folland 5.1.5(a,c,d)) Computing line integrals of vector fields*

-Explain briefly what a line integral is using the analogy of a boat sailing through a fixed wind pattern

To actually compute a line integral, you need two things:

(1) A vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$
(2) A parameterized path, $\gamma(t) : [a, b] \to \mathbb{R}^3$

At each point on the curve $\gamma(t)$, you project the vector field onto the direction of the curve at $\gamma(t)$, then sum the magnitude of this quantity over the whole curve. The magnitude of the projection at $\gamma(t)$ is exactly given by $\mathbf{F}(\gamma(t)) \cdot \gamma'(t)$. Let's do some examples; in the following, we will compute:

$$\int_C \mathbf{F} \cdot d\mathbf{x}$$

Part a) $\mathbf{F}(x, y, z) = (yz, x^2, xz)$ on C the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

- The first thing we need to do is parameterize the path.
- The direction of the path is given by $(1, 1, 1) - (0, 0, 0) = (1, 1, 1)$

$$\gamma(t) = (1, 1, 1)t \Rightarrow \gamma'(t) = (1, 1, 1)$$

- The curve starts at $t = 0$ and ends at $t = 1$
- Now we compute the magnitude of the component of $\mathbf{F}$ which points in the direction of the curve at the point $\gamma(t)$.

$$\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = (y(t)z(t), x(t)^2, x(t)z(t)) \cdot (1, 1, 1) = (t^2, t^2, t^2) \cdot (1, 1, 1) = 3t^2$$

- We finally integrate this over the whole parameterized curve:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 3t^2\, dt = 1$$

Part c) $\mathbf{F}(x, y) = (x - y, x + y)$, C is the unit circle $x^2 + y^2 = 1$ oriented clockwise

- The first thing we need to do is parameterize the path.
- What functions satisfy $x(t)^2 + y(t)^2 = 1$? Pick $x(t) = \cos t$ and $y(t) = \sin t$

$$\gamma(t) = (\cos t, \sin t) \Rightarrow \gamma'(t) = (-\sin t, \cos t)$$

- We want to traverse the unit circle *clockwise*, so the path should start at $t = 0$ and end at $t = -2\pi$
- Now we compute the magnitude of the component of $\mathbf{F}$ which points in the direction of the curve at the point $\gamma(t)$.

$$\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = (\cos t - \sin t, \cos t + \sin t) \cdot (-\sin t, \cos t) = -\sin t \cos t + \sin^2 t + \cos^2 t + \cos t \sin t = 1$$

- We finally integrate this over the whole parameterized curve:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{-2\pi} dt = -2\pi$$